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Selberg's Method and the Multiplicities of the Zeroes of the Riemann Zeta-Function

by

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Dedicated to Professor Akio Fujii who had the patience to listen to my presentation when he might have otherwise had lunch.

Abstract. This article shows how the approximation of $S(T)$ by a Dirichlet polynomial, work begun by Selberg, can be used to recover bounds on the multiplicities of the zeroes of $\zeta(s)$.

1. Introduction

This paper concerns the properties of the moments of the function $S(T)$, the argument of the Riemann zeta-function; for some basic properties the reader is invited to examine [9, Ch. 9]. The function $S(T)$ is intimately connected with $N(T)$, the number of non-trivial zeroes[1] of $\zeta(\sigma + it)$ for $0 < t < T$ through the following formula

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}), \tag{1}$$

where the $O(T^{-1})$ term is continuous. This continuity of the error term is essential, since it enables one to infer, by (1), that when T passes over a zero of multiplicity k, which is counted k times in $N(T)$, the function $S(T)$ increases by k. Results on the rate of growth of $S(T)$ therefore provide information as to the likelihood of zeroes of large multiplicity. Let $M(T)$ denote the maximum multiplicity of a zero between T and $2T$. Then

$$S(T) = O(\log T) \quad \Longrightarrow \quad M(T) = O(\log T), \tag{2}$$

and on the Riemann hypothesis

$$S(T) = O\left(\frac{\log T}{\log\log T}\right) \quad \Longrightarrow \quad M(T) = O\left(\frac{\log T}{\log\log T}\right). \tag{3}$$

Equation (2) is due to von Mangoldt and (3) is due to Littlewood (see, e.g. [9, p. 214 and p. 350]). This reliance of results on $S(T)$ to produces bounds for $M(T)$ should not be pushed too far. Selberg [8] showed that

$$S(T) = \Omega_{\pm}\left(\frac{(\log T)^{1/3}}{(\log\log T)^{7/3}}\right),$$

[1] Here, and hereafter, *zeroes* always refers to non-trivial zeroes of the zeta-function.

whereas it is believed that all the zeroes of the zeta-function should be simple. Nevertheless, one can pursue the connexion between moments of $S(T)$ and zeroes of high multiplicity. Throughout this paper, A denotes a positive constant, not necessarily the same at each occurrence.

2. Selberg's Approximation

Selberg proved the following theorem in [8, p. 39, Thm 4].

THEOREM 2.1 (Selberg). *If $T^{\alpha} < H \leq T$, where α is fixed and $\frac{1}{2} < \alpha \leq 1$, and, for m a positive integer, $T^{\frac{\alpha-\frac{1}{2}}{m}} \leq x \leq H^{\frac{1}{m}}$, then*

$$\int_{T}^{T+H} \left| S(t) + \frac{1}{\pi} \sum_{p<x} \frac{\sin(t \log p)}{\sqrt{p}} \right|^{2m} dt \leq c(m)H\,,$$

where $c(m)$ depends on m but not on T.

The factor $c(m)$ has been improved over the years: in Selberg's paper [*op. cit.*] it was not calculated explicitly. Fujii [3] calculated that $c(m) \leq (Am)^{4m}$, and this result, which was also proved by Ghosh [4], follows more or less directly from Selberg's original arguments. Tsang [11] showed that $c(m) \leq (Am)^{2m}$, where the improvement comes from a repeated application of Selberg's density theorem, specifically Lemma 5.2 of [10]. Karatsuba and Korolev [6] placed a bound on the explicit constant in the result of Tsang, to show that $c(m) < (\epsilon^{-3}e^{37}\pi^{-2}m^{2})^{m}$, for any fixed positive $\epsilon < 0.001$. A different approach is due to Goldston [5]. On the assumption of the Riemann hypothesis and introducing a different weight to that used by Selberg, Goldston showed that $c(m) \geq (A \log m)^{m}$.

3. The connexion with multiplicities

Define $N_j(T)$ to be the number of zeroes of multiplicity j at height T. Fujii [2] showed that

$$\frac{N_j(2T) - N_j(T)}{N(2T) - N(T)} \ll e^{-A\sqrt{j}}\,, \tag{4}$$

by using $c(m) \leq (Am)^{4m}$ in Theorem 2.1. Later Fujii [1] proved (4) with a bound of e^{-Aj}; an explicit version of this improvement was given by Korolev [7]. In general, if one can show that $c(m) \leq (Am)^{\alpha m}$ for some α, then one can show[2] that

$$\frac{N_j(2T) - N_j(T)}{N(2T) - N(T)} \ll \exp(-Aj^{2/\alpha})\,. \tag{5}$$

Thus if there is one zero of multiplicity j between T and $2T$ one can use (5) to show that

$$(T \log T)^{-1} \ll \exp(-Aj^{2/\alpha})\,,$$

[2] This is implicit in the work of Fujii [3]—I thank the referee for making this known to me.

so that

$$j \ll (\log T)^{\alpha/2} \ .$$

Thus Tsang's result (viz. $\alpha = 2$) recovers the best unconditional bound for $M(T)$. This shows that any improvement on the constant $c(m)$ is likely to be difficult since $c(m) \leq (Am)^{\alpha m}$ for $\alpha < 2$ would surpass that which is currently provable on the assumption of the Riemann hypothesis—cf. (3).

3.1. Concluding question

The problem: assume the Riemann hypothesis and improve the bound on $c(m)$ to show that

$$M(T) \ll \frac{\log T}{\log\log T} \ .$$

This, being (3), is already known, but it would be of interest to see how much could be recovered unconditionally using this new approach.

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